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Construction of the Thermodynamic Jamming Limit for the Parking Process and Other Exclusion Schemes on $\mathbb{Z}^d$

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We provide an explicit construction of the thermodynamic jamming limit for the parking process and other finite range exclusion schemes on $\mathbb{Z}^d$. By means thereof, a strong law of large numbers for occupation densities is accomplished, and, amongst other results, the so called "super-exponential" (i.e. gamma) decay of pair correlation functions is established.

KEY WORDS: Random sequential adsorption (RSA); Parking process; Pair correlation functions; Super-exponential decay; Thermodynamic limit; Perfect simulation; Multidimensional ergodic theorem.
Mathematics Subject Classification (1991): 60K99.

1. INTRODUCTION

In its simplest form the *car-parking process* on $\mathbb{Z}^d$ (also called *random sequential adsorption* (RSA) in physical sciences) may be (informally) described as follows:

> a site is chosen uniformly over the box $\Lambda_n^{(d)} = \{-n, \ldots, 0, \ldots, n\}^d$ (the target region) and is occupied (spin/state 1) provided all its nearest neighbours are vacant (spin/state 0). Once occupied a site remains so for ever. The process continues until all sites in $\Lambda_n^{(d)}$ are either occupied or have at least one of their nearest neighbours occupied (the so called *jamming limit*). Call P_n the terminal configuration of $\Lambda_n^{(d)}$.[2]

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[2] Throughout this paper we shall abuse terminology and use P_n to denote not only the terminal configuration/jamming limit, but also the process/algorithm leading to it. Since only terminal configurations will be taken into account, this identification is reasonable.

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This model as many of its generalizations (including longer range interactions and particles of different kinds) have long received attention in physical sciences, wherein issues like kinetics near the jamming limit, spatial correlations, asymptotics (as the target region increases to infinity) of the first moments of $|P_n|$ (the ultimate number of adsorpted particles, i.e. occupied sites), amongst others, have been studied by a series of non-rigorous theoretical methods, computer simulations and experiments.[6,17]

Until recently, mathematical results have been restricted to one-dimensional models[3,9,12–14,16] and to quasi-one-dimensional models.[1,2,7]

In 2002 M. D. Penrose[15] proved, amongst other results, a law of large numbers (L^p-convergence) for the sequence $(|P_n|/|\Lambda_n^{(d)}|)_{n\geq 0}$ for a large class of exclusion models (including those with finitely many spin states), treated therein as (coupled) finite interacting particle systems.

In this paper we construct in a common probability space (i.e. couple) a limit "parking" algorithm P, having the whole $\mathbb{Z}^d$ lattice as target region, together with the finite parking algorithms P_n, $n \geq 0$ in such a way that $P_n \to P$ almost surely. This construction being sufficiently general to cope with all sort of finite range (exclusion) interactions in a $\{0, 1\}$ *spin* state scenery. In this set up P may be interpreted as the jamming limit of a virtual parking process over the whole $\mathbb{Z}^d$ lattice. We call P the thermodynamic limit of the sequence $(P_n)_{n\geq 0}$, and its distribution μ the corresponding thermodynamic limit measure.

By means of this construction (Section 3), we prove a strong law of large numbers for the sequence $(|P_n|/|\Lambda_n^{(d)}|)_{n\geq 0}$, via the classical multiparameter ergodic theorem (Section 4); establish the super-exponential (i.e. gamma) decay of pair correlation functions (Subsection 5.1) and show that the convergence to the thermodynamic limit μ occurs super-exponentially fast (Subsection 5.2).

2. BASIC DEFINITIONS AND NOTATION

The following structures will be basic blocks in the forthcoming constructions. Therefore we take the opportunity to establish the notation to be used henceforth.

2.1. Probability and Measure Spaces

The random objects we shall deal with in this paper will be defined on a fundamental probability space (i)—the space of uniform random variables attached to each site of $\mathbb{Z}^d$—and assume values in the (measure) space of spin configurations over the $\mathbb{Z}^d$ lattice (ii). As construction is the realm of this paper, we make these spaces explicit:

(i) $((0,1)^{\mathbb{Z}^d}, \mathcal{B}_{(0,1)}^{\mathbb{Z}^d}, \lambda_{(0,1)}^{\mathbb{Z}^d})$, where $\mathcal{B}_{(0,1)}$ is the usual Borel σ-algebra in $(0,1)$, $\lambda_{(0,1)}$ is the Lebesgue (uniform) measure in $(0,1)$ and $\cdot^{\mathbb{Z}^d}$ means product in the appropiate sense;

(ii) $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$, where $\mathcal{B}_{\{0,1\}}$ is taken for the power set of $\{0,1\}$, $\mathcal{P}(\{0,1\}) \stackrel{\text{def}}{=} \{\{0,1\},\{0\},\{1\},\emptyset\}$.

2.2. Graphs

In what follows the notion of vicinity between sites of the $\mathbb{Z}^d$ lattice will be implemented by the graph $\mathcal{G}_\nu^{(d)} \stackrel{\text{def}}{=} (\mathbb{Z}^d, \mathcal{E}_\nu^{(d)})$, where $\mathcal{E}_\nu^{(d)}$ is taken for set of bonds joining distinct sites x, y in $\mathbb{Z}^d$, such that $|x-y|_{\sup} \stackrel{\text{def}}{=} \max_{1\le i\le d}|x_i - y_i| \le \nu$, for some distance $\nu \in \mathbb{N}^* \stackrel{\text{def}}{=} \mathbb{N}\setminus\{0\}$.

2.3. Parking Schemes

A subset $S_\nu^{(d)} \subset \{0,1\}^{\Lambda_\nu^{(d)}}$, where $\Lambda_\nu^{(d)} \stackrel{\text{def}}{=} \{-\nu,\dots,0,\dots,\nu\}^d$ is the box/ball with centre at the origin and radius ν in the $\mathbb{Z}^d$ lattice, such that $S_\nu^{(d)} \ni 0^{\Lambda_\nu^{(d)}}$ will be called a *parking scheme* (with radius ν of interaction) on the $\mathbb{Z}^d$ lattice.[3]

As it will become clear below, a *parking scheme* is nothing but the set of neighborhoods which allow a site to be occupied.

2.4. Parking Processes

For a given *parking scheme* $S_\nu^{(d)}$, we define the *parking process* on the box $\Lambda_n^{(d)}$ with null boundary condition, $(\Lambda_n^{(d)} S_\nu^{(d)})PP\,nbc$, to be the following (random) algorithm:

Step 1. set $P_n = 0^{\mathbb{Z}^d}$;
Step 2. choose a site $x \in \Lambda_n^{(d)}$ uniformly amongst the sites not chosen previously;
Step 3. if $\theta_x(P_n)|\Lambda_\nu^{(d)} \in S_\nu^{(d)}$, then $P_n(x) \leftarrow 1$;
Step 4. if there are points in $\Lambda_n^{(d)}$ not chosen yet, then go back to step 2, else stop the algorithm.

Above, for all $y \in \mathbb{Z}^d$, θ_y is the translation operator, i.e. for all $P \in \{0,1\}^{\mathbb{Z}^d}$, $\theta_y(P)(x) \stackrel{\text{def}}{=} P(x+y)$. And $P|\Lambda_\nu^{(d)}$ means the restriction of P to the box $\Lambda_\nu^{(d)}$, that is, $P|\Lambda_\nu^{(d)} \in \{0,1\}^{\Lambda_\nu^{(d)}}$ and for all $x \in \Lambda_\nu^{(d)}$, $P|\Lambda_\nu^{(d)}(x) \stackrel{\text{def}}{=} P(x)$.

[3] As it will become clear from the forthcoming definitions, we may suppose, without loss of generality, that $\eta(0^d) = 0$ for all $\eta \in S_\nu^{(d)}$. Henceforth we shall abuse notation and write 0 for 0^d, whenever no confusion may arise.

It should be clear that the algorithm described above generates a random element P_n in the measure space $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$ and therefore a corresponding probability measure is induced therein.

We put these ideas into a *probability setting* by means of

Definition 21. For a given *parking scheme* $S_\nu^{(d)}$, we define the parking process on the box $\Lambda_n^{(d)}$ with null boundary condition, $(\Lambda_n^{(d)} S_\nu^{(d)}) PPnbc$, to be the following algorithm:

For each $\omega \in (0,1)^{\mathbb{Z}^d}$,

Step 1. set $P_n(\omega) = 0^{\mathbb{Z}^d}$;
Step 2. choose $x \in \Lambda_n^{(d)}$ such that $\omega(x) = \inf\{\omega(\xi) : \xi \in \Lambda_n^{(d)}$ and ξ has not been chosen previously$\}$;
Step 3. if $\theta_x(P_n(\omega))|\Lambda_\nu^{(d)} \in S_\nu^{(d)}$, then $P_n(\omega)(x) \leftarrow 1$;
Step 4. if there are points in $\Lambda_n^{(d)}$ not chosen yet, then go back to step 2, else stop the algorithm.

The formal algorithm described above is probabilisticaly equivalent to the former (informal) algorithm in the sense that both induce the same probability measure μ_n in $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$. This equivalence being due to the fact that the minimum of an *iid* finite family of uniform random variables is uniformly distributed amongst them.

We call the output $P_n(\omega)$ the *jamming limit* of the parking process just described[4] and observe that P_n is a $\{0,1\}^{\mathbb{Z}^d}$-valued random element on $((0,1)^{\mathbb{Z}^d}, \mathcal{B}_{(0,1)}^{\mathbb{Z}^d}, \lambda_{(0,1)}^{\mathbb{Z}^d})$.

Finally, we remark that each P_n, $n \geq 0$, is well (i.e. uniquely) defined, except for a set of null probability; ambiguity might occur at ω's such that $\omega(x) = \omega(y)$ for at least two distinct sites $x, y \in \mathbb{Z}^d$. We shall return to this point later on, but, as it usually happens in probability theory, that should pose us no difficulty at all.

A natural question that arises at this point is the existence (or not) of a thermodynamic limit probability measure μ for the sequence $(\mu_n)_{n\geq 1}$ of probability measures on $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$ induced by the sequence $(P_n)_{n\geq 1}$ of random elements just defined. We shall delve into this question in the next section.

3. CONSTRUCTION OF THE THERMODYNAMIC LIMIT

In what follows we show that, for a given parking scheme $S_\nu^{(d)}$, there exists a unique probability measure μ on $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$ such that $\mu_n \Rightarrow \mu$.[5] We shall

[4] In the forthcoming text we shall frequently abuse terminilogy and use P_n to denote also the underlying parking process leading to it.
[5] The symbol "$\Rightarrow$" denotes convergence in distribution.

accomplish this task by means of constructing a random element/algorithm P : $((0,1)^{\mathbb{Z}^d}, \mathcal{B}_{(0,1)}^{\mathbb{Z}^d}, \lambda_{(0,1)}^{\mathbb{Z}^d}) \longrightarrow (\{0,1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$ such that $P_n \to P a.s.$[6]

For that purpose we set the following definitions:

Definition 31. A path of points in the graph $\mathcal{G}_\nu^{(d)}$ is a finite sequence $(x_i)_{0\le i\le n}, x_i \in \mathbb{Z}^d$; such that $|x_{i+1} - x_i|_{\sup} \le \nu, 0 \le i \le n-1$.

In what follows we shall say that the path $(x_i)_{0\le i\le n}$ has length n (rather than $n+1$), bearing in mind that n bonds participate in such a path.

Definition 32. A path $(x_i)_{0\le i\le n}$ is said to be decreasing (subject to $\omega \in (0,1)^{\mathbb{Z}^d}$), whenever the numerical sequence $(x_i(\omega))_{0\le i\le n}$ is (strictly) decreasing.[7] In this case we write $\downarrow_{(\omega)} (x_i)_{0\le i\le n}$, or simply $\downarrow (x_i)_{0\le i\le n}$, should ω be clear from the context.

It should be clear from Definition 32. that a decreasing path is always self-avoiding.

Definition 33. For $x, y \in \mathbb{Z}^d$ and $\omega \in (0,1)^{\mathbb{Z}^d}$, we say that x is influenced by y (or, equivalentely, that y influences x) subject to ω, if there exists a decreasing path $\downarrow_{(\omega)} (x_i)_{0\le i\le n}$, such that $x_0 = x$ and $x_n = y$. In this case we write $y \uparrow_{(\omega)} x$, or simply $y \uparrow x$, should ω be clear from the context.

We remark that the $\uparrow$ relation is both transitive and reflexive.

Definition 34. Given a subset $X \subset \mathbb{Z}^d$, we define the *armour* of X (subject to $\omega \in (0,1)^{\mathbb{Z}^d}$) to be the (random) subset

$$\mathcal{A}(X)(\omega) \stackrel{\text{def}}{=} \{y \in \mathbb{Z}^d : \exists x \in X : y \uparrow_{(\omega)} x\}$$

As before, we shall omit ω whenever no confusion may arise.

It is clear from Definition 34. that, for $X, Y \subset \mathbb{Z}^d$ and $\omega \in (0,1)^{\mathbb{Z}^d}$,

(i) $\mathcal{A}(X)(\omega) \supset X$;
(ii) $X = \biguplus_{i=1}^n X_i \Rightarrow \mathcal{A}(X)(\omega) = \bigcup_{i=1}^n \mathcal{A}(X_i)(\omega)$; [8]
(iii) $X \subset Y \Rightarrow \mathcal{A}(X)(\omega) \subset \mathcal{A}(Y)(\omega)$.

What may not be so clear is the following lemma.

Lemma 31. *For any finite set $X \subset \mathbb{Z}^d$, $\mathcal{A}(X)$ is (almost surely) finite.*

[6] Pointwise convergence regarding the product topology in $\{0,1\}^{\mathbb{Z}^d}$.
[7] $x_i(\omega)$ is taken for $\omega(x_i)$, which should be formally prefered.
[8] $\biguplus$ denotes disjoint union.

Proof: Suppose initially that $X = \{0\}$ and consider the event $A_n \stackrel{\text{def}}{=} (\mathcal{A}(X) \not\subset \Lambda_{n\nu}^{(d)})$, $n \in \mathbb{N}$. A_n amounts to saying that there exists $y \in \mathbb{Z}^d \setminus \Lambda_{n\nu}^{(d)}$ such that $y \uparrow 0$, that is, there exists a decreasing (and hence self-avoiding) path $(x_i)_{0\leq i\leq m}$, $m > n$, starting at 0 and ending outside the box $\Lambda_{n\nu}^{(d)}$. Thus A_n occurs only if there exists a decreasing path of length n, $(x_i)_{0\leq i\leq n}$ starting at 0. Now observe that the probability that an arbitrary (but fixed) self-avoiding path $(x_i)_{0\leq i\leq n}$ is decreasing is $1/(n+1)!$ and that the total number of self-avoiding paths starting at 0 with length n is no larger than $(2\nu+1)^{dn}$, to conclude (by subadditivity) that

$$\mathbb{P}(A_n) \leq \frac{(2\nu+1)^{dn}}{(n+1)!} \leq \frac{(2\nu+1)^{dn}}{n!} \tag{1}$$

Since the sequence $(A_n)_{n\geq 0}$ is decreasing and $\mathbb{P}(A_n) \to 0$ by (1), we must have $\mathbb{P}(\cap_{n=0}^{\infty} A_n) = 0$. Now observe that $\{\#\mathcal{A}(X) = \infty\} = \cap_{n=0}^{\infty} A_n$ to conclude that $\mathbb{P}(\{\#\mathcal{A}(X) < \infty\} = \{\#\mathcal{A}(X) = \infty\}^c) = 1$, that is $\mathcal{A}(\{0\})$ is almost surely finite.

Now make use of the stationarity of the product measure $\lambda_{(0,1)}^{\mathbb{Z}^d}$ to extent this result in the case of $X = \{x\}$ ($\forall x \in \mathbb{Z}^d$) and the desired result follows by subadditivity. □

In what follows it will be convinient to generalize *parking processes* on boxes (Definition 2.1) to parking processes on arbitrary finite subsets of $\mathbb{Z}^d$. This generalization does not involve any difficulty and is performed in the natural way.

Definition 35. Let X be a finite subset of $\mathbb{Z}^d$. For a given *parking scheme* $S_\nu^{(d)}$, we define the parking process on X with null boundary condition, $(XS_\nu^{(d)})PPnbc$, to be the following algorithm:

For each $\omega \in (0,1)^{\mathbb{Z}^d}$,

Step 1. set $P_X(\omega) = 0^{\mathbb{Z}^d}$;
Step 2. choose $x \in X$ such that $\omega(x) = \inf\{\omega(\xi) : \xi \in X \text{and } \xi \text{ has not been chosen previously}\}$;
Step 3. if $\theta_x(P_X(\omega))|\Lambda_\nu^{(d)} \in S_\nu^{(d)}$, then $P_X(\omega)(x) \leftarrow 1$;
Step 4. if there are points in X not chosen yet, then go back to step 2, else stop the algorithm.

As before, we call the output $P_X(\omega)$ the *jamming limit* of the parking process just defined and observe that P_X is a $\{0,1\}^{\mathbb{Z}^d}$-valued random element on $((0,1)^{\mathbb{Z}^d}, \mathcal{B}_{(0,1)}^{\mathbb{Z}^d}, \lambda_{(0,1)}^{\mathbb{Z}^d})$.

Now we are ready for the definition of the limit random element/algorithm P:

Definition 36. The limit random element/algorithm $P : ((0,1)^{\mathbb{Z}^d}, \mathcal{B}_{(0,1)}^{\mathbb{Z}^d}, \lambda_{(0,1)}^{\mathbb{Z}^d}) \to (\{0,1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$ is defined by

$$P(\omega)(x) \stackrel{\text{def}}{=} P_{\mathcal{A}(\{x\})(\omega)}(\omega)(x), \quad \forall x \in \mathbb{Z}^d$$

In words Definition 36. above means that for each $\omega \in (0,1)^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$, we determine the *armour* of the set $\{x\}$, $\mathcal{A}(\{x\})$, by means of Definition 3.4, construct/implement the parking process over $\mathcal{A}(\{x\})$ according to Definition 3.5 and then assign the value of the corresponding *jamming limit*$P_{\mathcal{A}(\{x\})}$ at site x to $P(\omega)(x)$. Of course a parking scheme $S_\nu^{(d)}$ is implicit in Definition 36.

Our next step is to prove that $P_n \to P a.s.$ For that purpose we shall need the following lemma:

Lemma 32. $P_n|\mathcal{A}(\{x\}) = P_{\mathcal{A}(\{x\})}|\mathcal{A}(\{x\})$ *in* $\{\Lambda_n \supset \mathcal{A}(\{x\})\}$.

In words Lemma 32 asserts that the (random) sets $\mathcal{A}(\{x\})$ are stable with respect to the sequence $(P_n)_{n\geq 1}$; in the sense that, once n becomes sufficiently large so that $\Lambda_n \supset \mathcal{A}(\{x\})$, the *jamming limits*$P_m$, $m \geq n$ are identical over $\mathcal{A}(\{x\})$.

In the proof of Lemma 32 the notions of inner and outer boundaries of a set $X \subset \mathbb{Z}^d$ will be employed. We make them clear by means of the next definition.

Definition 37. For each $\nu \geq 1$, the $\nu-$inner boundary of $X \subset \mathbb{Z}^d$ is defined to be the set

$$\delta_\nu^{(in)}(X) \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^d : x \in X \text{ and } \exists y \in \mathbb{Z}^d \setminus X \text{ such that } |x-y|_{\sup} \leq \nu\}$$

In the same fashion, the ν-outer boundary of $X \subset \mathbb{Z}^d$ is defined to be the set

$$\delta_\nu^{(out)}(X) \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^d : x \notin X \text{ and } \exists y \in X \text{ such that } |x-y|_{\sup} \leq \nu\}$$

For simplicity, the subscripts will be suppressed, whenever ν is understood from the context.

Remark *An immediate consequence of Definitions 3.4 and 3.7 is that $x(\omega) \leq y(\omega)$, whenever $x \in \delta^{(in)}(\mathcal{A}(X))$, $y \in \delta^{(out)}(\mathcal{A}(X))$ and $|x-y|_{\sup} \leq \nu$, for all $X \subset \mathbb{Z}^d$. That is, an armour has the noticeable property that in the interface with the outer enviroment its sites are numerically smaller than the* interacting *outer sites.*

Keeping in mind the remark above, we see that the outer boundary of $\mathcal{A}(\{x\})$ behaves like a null boundary condition around it when performing the *parking algorithm*P_n (provided Λ_n is large enough to encompass $\mathcal{A}(\{x\})$).

Now, recalling Definitions 2.1 and 3.5, we see that P_n and $P_{\mathcal{A}(\{x\})}$ must be identical over $\mathcal{A}(\{x\})$ and Lemma 32 is thus established.

Lemma 32 is the key to the main result of this section:

Theorem 33. *(Almost sure convergence to the limit algorithm P).*

$$P_n \to P \quad \lambda_{(0,1)}^{\mathbb{Z}^d} a.s.$$

Remark *Before proving Theorem 33 we emphasize that the a.s. convergence therein is relative to the usual product topology of* $\{0, 1\}^{\mathbb{Z}^d}$.

Proof: Bearing in mind the foregoing remark, it is enough to show that for $\lambda_{(0,1)}^{\mathbb{Z}^d}$-almost all $\omega \in (0, 1)^{\mathbb{Z}^d}$ and every $m \geq 0$, $P_n(\omega)|\Lambda_m^{(d)} = P(\omega)|\Lambda_m^{(d)}$, for n sufficiently large.

Now, given $m \geq 0$, take n large enough so that $\mathcal{A}(\Lambda_m^{(d)}) \subset \Lambda_n^{(d)}$. This is possible for almost all ω in view of Lemma 31. For each $x \in \Lambda_m^{(d)}$, $\mathcal{A}(\{x\}) \subset \mathcal{A}(\Lambda_m^{(d)})$ (remark after Definition 3.4). Hence

$$P(x) \overset{def.36.}{=} P_{\mathcal{A}(\{x\})}(x) \overset{lem.32}{=} P_n(x), \forall x \in \Lambda_m^{(d)}$$

and we are done. □

Since the random elements $P_n, n \geq 0$ and P take values in a Polish measure space,[9], viz $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$ almost sure convergence implies weak convergence of the corresponding distributions therein [11, prop.9 pag.355]. Thus the following corollary:

Corollary 34. *If* $(\mu_n)_{n\geq 0}$ *and* μ *are the corresponding probability distributions of* $(P_n)_{n\geq 0}$ *and* P *in* $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$*, then* $\mu_n \Rightarrow \mu$.

Before finishing this section we wish to emphasize that the construction of the limit measure carried out in Definition 3.6 endows it with two noticeable properties, viz stationarity (i.e. translation invariance) and ergodicity.

In order to formalize this point a little further we observe that the random elements $(P_n)_{n\geq 0}$ and P just defined (Definitions 2.1 and 3.6 respectively) are well defined, except for a set of probability zero, viz the set of ω's in $(0, 1)^{\mathbb{Z}^d}$ that assign a same numerical value to at least two distinct sites of $\mathbb{Z}^d$ or on which an infinite descending path occurs.

[9] A measure space whose σ-algebra is generated by a topology which is metrizable, complete and separable.

This difficulty may be easily removed by setting $P_n = P = 0^{\mathbb{Z}^d}$, $n \geq 0$, for instance, over this troublesome set. Once this is done, we can define the real measurable function f on $((0, 1)^{\mathbb{Z}^d}, \mathcal{B}_{(0,1)}^{\mathbb{Z}^d}, \lambda_{(0,1)}^{\mathbb{Z}^d})$ by $f(\omega) \stackrel{def}{=} P(\omega)(0)$ and observe that $P(\omega)(x) = f(\theta_x(\omega))$, for all x in $\mathbb{Z}^d$. Since the product measure $\lambda_{(0,1)}^{\mathbb{Z}^d}$ is ergodic and stationary in $((0, 1)^{\mathbb{Z}^d}, \mathcal{B}_{(0,1)}^{\mathbb{Z}^d})$, so will be μ, the distribution of P, in $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$. This is a standard argument in ergodic theory and we refer the reader to [5, chapt.6] for further details.

Therefore we have established another corollary to theorem 33:

Corollary 35. *The thermodynamic limit measure* μ *is both stationary and ergodic in* $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d})$.

The construction of the thermodynamic limit measure μ carried out in this section corresponds to the concept of *perfect simulation* in the sense of Fernandez, Ferrari and Garcia.[8] In this technique, with the aid of a finite (yet random) number of i.i.d. (uniform) random variables, one generates (i.e. simulates) according to a prescribed thermodynamic measure a finite portion (window) of an infinite random object (field).

Besides the straightforward advantages comparing to ordinary *Monte Carlo* simulations, wherein only finite approximations of the thermodynamic measure are actually simulated, this constructive technique is also very useful regarding theoretical proofs as the forthcoming sections testify.

4. A STRONG LAW FOR OCCUPATION DENSITIES

First of all we formalize the concept of occupation density refered to in Section 1:

Definition 41. For $n \in \mathbb{N}$, we define the *occupation density* at the *jamming limit* P_n to be the random variable

$$\rho_n(\omega) = \frac{\sum_{x \in \Lambda_n^{(d)}} P_n(\omega)(x)}{|\Lambda_n^{(d)}| = (2n+1)^d} \quad on\big((0, 1)^{\mathbb{Z}^d}, \mathcal{B}_{(0,1)}^{\mathbb{Z}^d}, \lambda_{(0,1)}^{\mathbb{Z}^d}\big)$$

As we mentioned in Section 1, the existence (or not) of $\lim_{n\to\infty} \mathbb{E}(\rho_n)$ is a question of relevance in *random sequential adsorption* (RSA) contexts. Untill recently an (affirmative) answer was known only for one-dimensional models[9,12–14,16] or for quasi-one-dimensional models.[1,2,7] Recently Penrose[15] proved a law of large numbers (convergence in probability) for the sequence $(\rho_n)_{n\geq 1}$ for a large class of exclusion schemes (considered therein as finite particle systems) on $\mathbb{Z}^d$. This class overlaps to some extend the class considered here.[10] Naturally, this result assigns an affirmative answer (by means of a straightforward application of the Bounded Convergence Theorem) to the existence of $\lim_{n\to\infty} \mathbb{E}(\rho_n)$.

Here with the aid of the construction carried out in Section 3, we shall accomplish a simple proof for a strong law of large numbers (SLLN) for $(\rho_n)_{n\geq 1}$.

The first step to this proof is to observe that we can invoke *Wiener's multiparameter ergodic theorem* [10, Appendix 14.A] to conclude that

$$\lim_{n\to\infty} \frac{\sum_{x\in\Lambda_n^{(d)}} P(\omega)(x)}{|\Lambda_n^{(d)}|} = \lim_{n\to\infty} \frac{\sum_{x\in\Lambda_n^{(d)}} f\circ\theta_x(\omega)}{(2n+1)^d} = \mathbb{E}(f)$$

$$= \mathbb{E}(f\circ\theta_0) = \mathbb{E}(P(0)) \stackrel{\text{def}}{=} \rho \qquad \lambda_{(0,1)}^{\mathbb{Z}^d}\text{-a.s.} \tag{2}$$

for some constant $\rho \in [0, 1]$. This is true since f is measurable and integrable on $((0,1)^{\mathbb{Z}^d}, \mathcal{B}_{(0,1)}^{\mathbb{Z}^d}, \lambda_{(0,1)}^{\mathbb{Z}^d})$ and the product measure is stationary and ergodic in $((0,1)^{\mathbb{Z}^d}, \mathcal{B}_{(0,1)}^{\mathbb{Z}^d})$.

One might guess that the discrepancies between $P(\omega)|\Lambda_n^{(d)}$ and $P_n(\omega)|\Lambda_n^{(d)}$ are due to some kind of boundary effect. Next lemma quantifies the extension of this boundary pertubation.

Lemma 41. *If A_n is the event $\{\mathcal{A}(\Lambda_n^{(d)}) \not\subset \Lambda_{(n+\lfloor\sqrt{n}\rfloor\nu)}^{(d)}\}$, then $\mathbb{P}(\limsup A_n) = 0$, i.e. with probability one the sequence $(A_n)_{n\geq 1}$ occurs finitely many times.*[11]

Proof: By means of the estimation (1) carried out in Lemma 31 and bearing in mind the stationarity of the product measure, subadditivity yields:

$$\mathbb{P}(A_n) \leq (2n+1)^d \cdot \frac{(2\nu+1)^{d\lfloor\sqrt{n}\rfloor}}{\lfloor\sqrt{n}\rfloor!} := a_n$$

Therefore by Stirling's formula[12] :

$$\mathbb{P}(A_n) \leq a_n \simeq \frac{1}{\sqrt{2\pi}} \cdot \frac{(2n+1)^d}{\lfloor\sqrt{n}\rfloor^{1/2}} \cdot \frac{(2\nu+1)^{d\lfloor\sqrt{n}\rfloor}}{\lfloor\sqrt{n}\rfloor^{\lfloor\sqrt{n}\rfloor}} \cdot e^{\lfloor\sqrt{n}\rfloor} := b_n$$

But, since $b_n = o(1/n^2)$, [13]

$$\sum_{n=1}^{\infty} b_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty \Rightarrow \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty.$$

[10] We refer the reader to Section 6 for a discussion about this point.

[11] $\lfloor x\rfloor \stackrel{\text{def}}{=}$ greatest integer less than or equal to x.

[12] $n! \simeq n^n e^{-n}\sqrt{2\pi n}$, i.e. $n^n e^{-n}\sqrt{2\pi n}/n! \to 1$ as $n\to\infty$.

[13] Landau's notation: $f_n = o(g_n)$ means that $f_n/g_n \to 0$ as $n\to\infty$.

Thus the proof is concluded by an application of the first Borel-Cantelli lemma. □

Now we are ready for the main result of this section:

Theorem 42. *A Strong Law of Large Numbers for Occupation Densities:*

$$\rho_n \to \rho \quad \lambda^{\mathbb{Z}^d}_{(0,1)}\text{-}a.s.$$

Proof: We know by Lemma 41 that the last occurrence of $(A_n)_{n\geq 1}$ is almost surely finite, that is, $\bar{n} = \sup\{n \in \mathbb{N} : A_n \text{ occurs}\} < \infty\, a.s.$ Given $\epsilon > 0$, pick $i \in \mathbb{N}$ sufficiently large so that $i \in [n + \lfloor\sqrt{n}\rfloor\nu, (n+1) + \lfloor\sqrt{n+1}\rfloor\nu)$ for some $n > \bar{n}$ and such that $(\frac{2(m+1)+\lfloor\sqrt{m+1}\rfloor\cdot 2\nu+1}{2m+1})^d - 1 < \epsilon$, for all $m \geq n$. Therefore $P|\Lambda_n^{(d)} = P_i|\Lambda_n^{(d)}$ by the Proof of Theorem 33.

Hence

$$\left|\frac{\sum_{x\in\Lambda_i^{(d)}} P(x) - \sum_{x\in\Lambda_i^{(d)}} P_i(x)}{(2i+1)^d}\right| \leq \frac{\{[(n+1) + \lfloor\sqrt{n+1}\rfloor\nu]\cdot 2 + 1\}^d - (2n+1)^d}{(2n+1)^d}$$

$$= \left(\frac{2(n+1) + \lfloor\sqrt{n+1}\rfloor\cdot 2\nu + 1}{(2n+1)}\right)^d - 1 < \epsilon$$

That is

$$\lim_{i\to\infty}\left(\frac{\sum_{x\in\Lambda_i^{(d)}} P(x)}{(2i+1)^d} - \frac{\sum_{x\in\Lambda_i^{(d)}} P_i(x)}{(2i+1)^d}\right) = 0 \quad \text{a.s.}$$

Since $\lim_{i\to\infty}\frac{\sum_{x\in\Lambda_i^{(d)}} P(x)}{(2i+1)^d} = \rho\, a.s.$ by (2), $\rho_i = \frac{\sum_{x\in\Lambda_i^{(d)}} P_i(x)}{(2i+1)^d} \to \rho\, a.s.$ □

As we mentioned in the beginning of this section, a straightfoward application of the Bounded Convergence Theorem now garantees the existence of $\lim_{n\to\infty}\mathbb{E}(\rho_n)$.

Corollary 43. $\lim_{n\to\infty}\mathbb{E}(\rho_n) = \rho$

5. SUPER-EXPONENTIAL BEHAVIOUR

First of all we precise the meaning of *super-exponential decay*.

Definition 51. A function $f : \mathbb{N} \to \mathbb{R}$ is said to decay *super-exponentially* whenever $f(n) = o(e^{-\alpha n})$, $\forall\alpha > 0$, i.e. $\lim_{n\to\infty} f(n).e^{\alpha n} = 0$, $\forall\alpha > 0$.

In physicochemical literature the so called *random sequential adsorptian* (RSA) processes are distinguished from *equilibrium systems* by several noticeable properties.[6,17] Amongst them we pay special attention to the super-exponential decay of pair correlation functions, a feature of RSA models that differs remarkably from the (usual) exponential decay of equilibrium systems. Although described in physical and physicochemical literature,[6,17] a rigorous account thereof is available only for one-dimensional models.[3]

Here super-exponential decay of pair-correlation functions is established from the construction of the thermodynamical limit measure μ carried out in Section 3. This amounts to saying that this feature is a universal property of the whole class of exclusion schemes studied in this paper, regardless of dimension (d) and range of interaction (ν).

Another universal property derived from Section 3 is the super-exponential *speed* of convergence to the limit measure μ. We cope with this point in Subsection 5.2

5.1. The Super-Exponential Decay of Pair-correlation Functions

We recall that the *correlation coefficient* $\sigma_{X,Y}$ of two random variables X, Y is defined by $\sigma_{X,Y} = \mathrm{Cov}(X, Y)/(\sigma_X \sigma_Y)$, where $\mathrm{Cov}(X, Y)$ stands for the covariance of X and Y, and σ_X, σ_Y for the corresponding standard deviations.

Regarding the exclusion schemes studied here, one might guess how the state of site $x \in \mathbb{Z}^d$ influences the state of another site $y \in \mathbb{Z}^d$ with respect to the limit measure μ. Therefore we shall investigate the asymptotic behaviour of the correlation coefficient σ_{xy} of the random variables $P_x \overset{\text{def}}{=} P(x)$ and $P_y \overset{\text{def}}{=} P(y)$ as the distance $|x - y|_{sup}$ tends to infinity. Thanks to the translation invariace of μ (Corollary 3.5) $\sigma_{xy} = \sigma_{0(y-x)}$, and we may suppose $x = 0$ without loss of generality.

The central result of this subsection (Theorem 51) asserts that σ_{0x} decays super-exponentially as x tends towards infinity. More precisely, this amounts to saying that $\lim_{x\to\infty} \sigma_{0x} e^{\alpha|x|} = 0$, $\forall\alpha > 0$. In order to prove Theorem 51, it will be convenient to enlarge (i.e. square) our basic probability space $((0, 1)^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}_{(0,1)}, \lambda^{\mathbb{Z}^d}_{(0,1)})$ to $([(0, 1)^2]^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}_{(0,1)^2}, \lambda^{\mathbb{Z}^d}_{(0,1)^2})$. In this new setting each $\tilde{\omega} \in [(0, 1)^2]^{\mathbb{Z}^d}$ may be thought of as an ordered pair $\tilde{\omega} = (\omega_1, \omega_2)$, for $\omega_i \in (0, 1)^{\mathbb{Z}^d}$, $i = 1, 2$.

On this squared probability space we define the following projection operators taking values in the former unsquared measure space $((0, 1)^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}_{(0,1)})$:

Definition 52.

Cardinal Projection Operators:

$$\omega_1(\tilde{\omega}) \stackrel{\text{def}}{=} \omega_1; \quad \omega_2(\tilde{\omega}) \stackrel{\text{def}}{=} \omega_2.$$

Hybrid Projection Operators:

$$\omega_{i,j}^{(n)}(\tilde{\omega})(x) \stackrel{\text{def}}{=} \omega_i(\tilde{\omega})(x)1_{\{x_1 \le n\}} + \omega_j(\tilde{\omega})(x)1_{\{x_1 > n\}}; \quad i, j \in \{1, 2\}, \quad i \ne j,$$

for $n \in \mathbb{N}$ and bearing in mind the canonical representation $x = (x_1, x_2, \ldots, x_d)$, for sites in $\mathbb{Z}^d$ as d-tuples of integers.[14]

In words Definition 5.2 denotes that $\omega_{i,j}^{(n)}$ plays the role of ω_i in the half hyper-space $\{x_1 \le n\}$ and the role of ω_j in $\{x_1 > n\}$.

A straightfoward implication of Definition 5.2 is that all the projection operators above share the same probability distribution in $\mathcal{B}_{(0,1)}^{\mathbb{Z}^d}$, viz $\lambda_{(0,1)}^{\mathbb{Z}^d}$. Moreover, $\omega_1 \perp \omega_2$ and $\omega_{1,2}^{(n)} \perp \omega_{2,1}^{(n)}$, $\forall n \ge 0$.[15]

The foregoing paragraph amounts to saying that $P(\omega_1), P(\omega_2), P(\omega_{1,2}^{(n)}), P(\omega_{2,1}^{(n)})$ share the same probability distribution μ in $\mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d}$ and that $P(\omega_{1,2}^{(n)})$ and $P(\omega_{2,1}^{(n)})$ are independent, i.e.

$$P(\omega_1) \sim P(\omega_2) \sim P\left(\omega_{1,2}^{(n)}\right) \sim P\left(\omega_{2,1}^{(n)}\right) \sim \mu, \forall n \ge 0 \tag{3}$$

$$P\left(\omega_{1,2}^{(n)}\right) \perp P\left(\omega_{2,1}^{(n)}\right), \forall n \ge 0 \tag{4}$$

Now we are prepared to prove the key result of this subsection, viz[16]

Theorem 51. *(Super-exponential decay of correlation functions). σ_{0x} decays super-exponentially as $x \to \infty$, i.e.* $\lim_{x\to\infty} \sigma_{0x} \cdot e^{\alpha|x|} = 0, \forall \alpha > 0$.

Proof: Let $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$ and suppose initially that $x_1 = |x| = \sup_{1\le i\le d}\{|x_i|\}$. Set $y = \lfloor x_1/2 \rfloor$ and pick the largest $n \in \mathbb{N}$ such that $2\nu n + 4\nu \le x_1$, i.e. $n = \lfloor x_1/2\nu \rfloor - 2$. Hence[17] $n\nu + 2\nu \le y \le |x| - (n\nu + 2\nu)$. Now observe that $\mathcal{A}(\{0\})(\omega_1) \subset B(0, n\nu) \Rightarrow \mathcal{A}(\{0\})(\omega_1) = \mathcal{A}(\{0\})(\omega_{1,2}^{(y)})$. This is true because ω_1 and $\omega_{1,2}^{(y)}$ are indistinguishable on the left of y (Definition 5.2). By the same token $\mathcal{A}(\{x\})(\omega_1) \subset B(x, n\nu) \Rightarrow \mathcal{A}(\{x\})(\omega_1) = \mathcal{A}(\{x\})(\omega_{2,1}^{(y)})$. Thus on the event $E \in \mathcal{B}_{(0,1)^2}^{\mathbb{Z}^d}$,

$$E \stackrel{\text{def}}{=} (\mathcal{A}(\{0\})(\omega_1) \subset B(0, n\nu)) \cap (\mathcal{A}(\{x\})(\omega_1) \subset B(x, n\nu))$$

---

[14] $1_{\{x_1 \le n\}} = 1$ if $x_1 \le n$, and 0 otherwise ...

[15] $\perp$ stands for independence.

[16] Henceforth, for $x \in \mathbb{Z}^d$, $|x|$ stands for $|x|_{\sup}$.

[17] Remember that ν stands for the *radius of interaction* of the underlying parking scheme $S_\nu^{(d)}$.

we must have

$$P(\omega_1)(0) \overset{\text{def.36.}}{=} P_{\mathcal{A}(\{0\})(\omega_1)}(\omega_1)(0) = P_{\mathcal{A}(\{0\})(\omega_{1,2}^{(y)})}(\omega_{1,2}^{(y)})(0) \overset{\text{def.36.}}{=} P(\omega_{1,2}^{(y)})(0)$$

$$P(\omega_1)(x) \overset{\text{def.36.}}{=} P_{\mathcal{A}(\{x\})(\omega_1)}(\omega_1)(x) = P_{\mathcal{A}(\{x\})(\omega_{2,1}^{(y)})}(\omega_{2,1}^{(y)})(x) \overset{\text{def.36.}}{=} P(\omega_{2,1}^{(y)})(x)$$

and hence

$$P(\omega_1)(0) \cdot P(\omega_1)(x) = P(\omega_{1,2}^{(y)})(0) \cdot P(\omega_{2,1}^{(y)})(x) \quad \text{on } E. \tag{5}$$

Since

$$\mathbb{P}(E^c) \leq 2 \cdot \frac{(2\nu+1)^{d \cdot n}}{n!} \tag{6}$$

by (1); (5) and (6) yield

$$\left|\mathbb{E}[P(\omega_1)(0) \cdot P(\omega_1)(x)] - \mathbb{E}\left[P\left(\omega_{1,2}^{(y)}\right)(0) \cdot P\left(\omega_{2,1}^{(y)}\right)(x)\right]\right| \tag{7}$$

$$=$$

$$\left|\mathbb{E}\left[P(\omega_1)(0) \cdot P(\omega_1)(x) - P\left(\omega_{1,2}^{(y)}\right)(0) \cdot P\left(\omega_{2,1}^{(y)}\right)(x)\right]\right|$$

$$\leq \mathbb{E}\left(\left|([P(\omega_1)(0) \cdot P(\omega_1)(x) - P\left(\omega_{1,2}^{(y)}\right)(0) \cdot P\left(\omega_{2,1}^{(y)}\right)(x)\right|\right)$$

$$\leq \mathbb{P}(E^c) \leq 2 \cdot \frac{(2\nu+1)^{d \cdot n}}{n!}$$

However

$$\begin{aligned}\mathbb{E}\left[P\left(\omega_{1,2}^{(y)}\right)(0) \cdot P\left(\omega_{2,1}^{(y)}\right)(x)\right] &= \mathbb{E}\left[P\left(\omega_{1,2}^{(y)}\right)(0)\right] \cdot \mathbb{E}\left[P(\omega_{2,1}^{(y)})(x)\right] \\ &= \mathbb{E}[P(\omega_1)(0)] \cdot \mathbb{E}[P(\omega_1)(x)] \end{aligned} \tag{8}$$

where we have used (4) in the former equality and (3) in the latter. Thus substituting (8) in (7) yield

$$\left|\mathbb{E}[P(\omega_1)(0) \cdot P(\omega_1)(x)] - \mathbb{E}[P(\omega_1)(0)] \cdot \mathbb{E}[P(\omega_1)(x)]\right| \leq 2 \cdot \frac{(2\nu+1)^{d \cdot n}}{n!} \tag{9}$$

Meanwhile, $P \circ \omega_1$ (on the enlarged/squared probability space) and P (on the former/unsquared probability space) share the same distribution μ in $\mathcal{B}_{\{0,1\}}^{\mathbb{Z}^d}$ thanks to the remark after Definition 5.2, i.e. $(P \circ \omega_1) \sim P \sim \mu$. Therefore (9) amounts to

$$\left|\mathbb{E}(P_0 \cdot P_x) - \mathbb{E}(P_0) \cdot \mathbb{E}(P_x)\right| \leq 2 \cdot \frac{(2\nu+1)^{d \cdot n}}{n!} \tag{10}$$

Substituting n in terms of $|x|$ as in the first paragraph of the current proof, and recalling the definition of covariance of two random variables, we finally have

$$|\text{Cov}(P_0, P_x)| \leq 2 \cdot \frac{(2\nu + 1)^{d \cdot (\lfloor |x|/2\nu \rfloor - 2)}}{(\lfloor |x|/2\nu \rfloor - 2)!} \tag{11}$$

At this point we observe that nothing would change in the foregoing calculations if it were the case that $x_i = \pm|x|, 1 \leq i \leq d$, in the beginning of this proof. A suitable redefinition of the projection operators $\omega_{1,2}^{(\cdot)}$ and $\omega_{2,1}^{(\cdot)}$ along the corresponding coordinate axes being all we would have to do to establish (11) once again. Therefore (11) remains true for all $x \in \mathbb{Z}^d$.

Having said this, since

$$\sigma_0^2 \stackrel{\text{def}}{=} \text{Var}(P_0) = \text{Var}(P_x) \stackrel{\text{def}}{=} \sigma_x^2, \quad \forall x \in \mathbb{Z}^d$$

thanks to the stationarity of μ (Corollary 33), we simply make use of the definition of correlation coefficient in (11) to establish that

$$|\sigma_{0x}| \leq \frac{2}{\sigma_0^2} \cdot \frac{(2\nu + 1)^{d \cdot (\lfloor |x|/2\nu \rfloor - 2)}}{(\lfloor |x|/2\nu \rfloor - 2)!} = o(e^{-\alpha|x|}), \quad \forall \alpha > 0 \tag{12}$$

and hence conclude the proof. □

5.2. Speed of Convergence to the Thermodynamic Limit

Since P_n actually takes values in the finite set $\{0, 1\}^{\Lambda_n^{(d)}}$, it is reasonable to quantify the rate at which the sequence $(\mu_n)_{n \geq 0}$ converges to the limit measure μ by estimating how fast the numerical sequence $|\mu(L) - \mu_n(L)|$ converges to zero, whenever L is a *local* set, i.e. a set determined by a finite number of sites in $\mathbb{Z}^d$.

The concept of *locallity* can be formalized as follows:

Definition 53. A set $L \in \mathcal{B}_{\{0,1\}^{\mathbb{Z}^d}}$ is said to be local (or cylindrical) if for some $m \in \mathbb{N}$ there exists a subset $\tilde{L} \subset \{0, 1\}^{\Lambda_m^{(d)}}$ such that

$$\{0, 1\}^{\mathbb{Z}^d} \ni \omega \in L \Leftrightarrow \{0, 1\}^{\Lambda_m^{(d)}} \ni \omega|\Lambda_m^{(d)} \in \tilde{L}$$

We write $\mathcal{L}$ for the class of all local sets in $\mathcal{B}_{\{0,1\}^{\mathbb{Z}^d}}$.

The next theorem asserts that the numerical convergence above is super-exponentially fast. Since P and P_n differ over the box Λ_m ($m < n$) only if $\mathcal{A}(\Lambda_m) \not\subset \Lambda_n$ (by the proof of Theorem 33) and this event occurs with a super-exponentially small probability (as $n \to \infty$), this result is not surprising and the theorem is stated without proof.

Theorem 52. *If $L \in \mathcal{L}$, then $|\mu(L) - \mu_n(L)| = o(e^{-\alpha n})$, $\forall \alpha > 0$.*

6. FINAL REMARKS

6.1. Parking Processes as Interacting Particle Systems[18]

One might think of the parking processes P_n, $n \geq 0$[19] described in Definition 2.1 as discrete-time versions (i.e. skeletons) of "finite" interacting particle systems $(\eta_t^{(n)})_{t\geq 0}$ with initial condition $\eta_0 = 0^{\mathbb{Z}^d}$ and *flip* rates $c(x, \eta)$ given by

$$c(x,\eta) = 1_{\{x\in\Lambda_n^{(d)}\}} \cdot 1_{\{\theta_x(\eta)|\Lambda_\nu^{(d)}\in S_\nu^{(d)}\}} \quad \forall x \in \mathbb{Z}^d, \eta \in \{0,1\}^{\mathbb{Z}^d}$$

It is not difficult to see that this is indeed the case, provided $S_\nu^{(d)}$ is a decreasing set[20]. (The foregoing correspondence being not correct otherwise, since non-decreasing parking schemes generate non-markovian parking processes.)

In this setting it would be natural to define the properly infinite interacting particle system $(\eta_t)_{t\geq 0}$, defined by the *flip* rates

$$c(x,\eta) = 1_{\left\{\theta_x(\eta)|\Lambda_\nu^{(d)}\in S_\nu^{(d)}\right\}} \quad \forall x \in \mathbb{Z}^d, \eta \in \{0,1\}^{\mathbb{Z}^d}$$

and same initial condition $\eta_0 = 0^{\mathbb{Z}^d}$ and construct (i.e. couple) $(\eta_t^{(n)})_{t\geq 0}, n \geq 0$ and $(\eta_t)_{t\geq 0}$ together making use of the same *Poisson marks* to conclude that

$$\lim_{t\to\infty} \eta_t^{(n)} \stackrel{\text{def}}{=} \eta_\infty^{(n)} \xrightarrow{(n)} \eta_\infty \stackrel{\text{def}}{=} \lim_{t\to\infty} \eta_t \quad \text{a.s.}$$

Since the sequence $(\eta_\infty^{(n)})_{n\geq 0}$ is probabilisticaly indistinguishable from the the sequence of jamming limits $(P_n)_{n\geq 0}$ from the foregoing sections (i.e. $(\eta_\infty^{(n)})_{n\geq 0} \sim (P_n)_{n\geq 0}$), we may interpret η_∞ as P, "the limit jamming limit" of Section 3 (Definition 3.6).

All this reasoning is meaningful and in fact it proves in straghtfoward fashion the existence of the limit measure μ (at least for the markovian case). Yet one should bear in mind that the proofs carried out in Sections 4 and 5 rely on the specifics of Section 3. Put it in other terms, the sole existence of μ is not enough to accomplish the results of this paper.

6.2. Parking Processes with Boundary Conditions Other than Null

It is easy to see that whatever boundary condition we use in Definition 2.1 we always obtain the same thermodynamic jamming limit P of Definition 3.6. More

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[18] We refer the interested reader to R. Durrett's review article[14] for a quick, yet meaningful, introduction to this subject.

[19] As we already paid attention to, we are abusing terminology and using the jamming limit P_n to denote the underlying parking process.

[20] i.e. for $\xi_1, \xi_2 \in S_\nu^{(d)}$, if $\xi_1 \preccurlyeq \xi_2$, then $\xi_2 \in S_\nu^{(d)} \Rightarrow \xi_1 \in S_\nu^{(d)}$; where '$\preccurlyeq$' is taken for the usual partial order of *spin* configurations.

precisely, if the sequence $(\tilde{P}_n)_{n\geq 1}$ is defined in the same probability space of the former sequence $(P_n)_{n\geq 1}$; exactly as prescribed in Definition 2.1, with the sole exception that we modify step(i) to

$$\textbf{Step}(\tilde{\textbf{i}}) \qquad \tilde{P}_n|\Lambda_n^{(d)} \leftarrow 0^{\Lambda_n^{(d)}}; \qquad \tilde{P}_n|\left(\mathbb{Z}^d \setminus \Lambda_n^{(d)}\right) \leftarrow \eta|\left(\mathbb{Z}^d \setminus \Lambda_n^{(d)}\right)$$

for an arbitrary $\eta \in \{0, 1\}^{\mathbb{Z}^d}$, then

$$\tilde{P}_n \longrightarrow P \longleftarrow P_n \quad \text{a.s.}$$

To see this we only have to modify Lemma 32 slightly as follows:

> For each $\omega \in \{0, 1\}^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$, if $\Lambda_n \supset \mathcal{A}(\{x\})(\omega)$, then $P_{n+\nu}$ and $P_{\mathcal{A}(\{x\})}$ are identical over $\mathcal{A}(\{x\})$; that is $P_{n+\nu}|\mathcal{A}(\{x\}) = P_{\mathcal{A}(\{x\})}|\mathcal{A}(\{x\})$; and to take n sufficiently large in Theorem 33 so that $\mathcal{A}(\Lambda_m^{(d)}) \subset \Lambda_{n-\nu}^{(d)}$.

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